Kesten-Stigum theorem for a supercritical multi-type branching process in a random environnement

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Summary

Summary

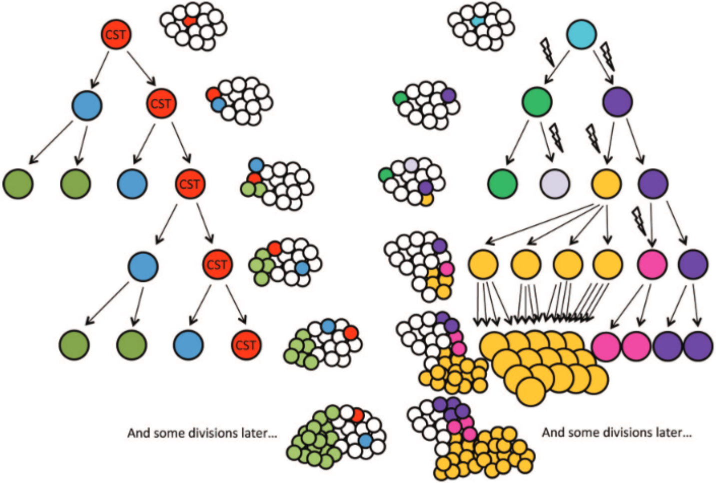
Background

- Smith and Wilkinson (1969) : i.i.d. environment, extinction.
- Athreya and Karlin (1971) : stationary and ergodic environment, fundamental limit theorems.
- Critical and subcritical cases : survival probability and conditional limit theorems ($d \geq 1$), see e.g. Vatutin & Dyakonova (2020, 2018), Vatutin & Wachtel (2018), Le Page, Peigné & Pham (2018) for $d > 1$, Afanasyev, Böinghoff, Kersting & Vatutin (2014, 2012), Afanasyev, Geiger, Kersting & Vatutin (2005) for $d = 1$.
- Supercritical case : large deviations ($d = 1$), see e.g. Buraczewski & Dyszewski (2020), Grama, Liu & Miqueu (2017), Bansaye & Böinghoff (2014, 2013, 2011), Huang & Liu (2012), Bansaye & Berestycki (2009).

Here we focus on the supercritical case with $d > 1$, and search for asymptotic properties.

Summary

Multi-type branching process



Multi-type Galton-Watson process

A d-type branching process $Z_n^i = (Z_n^i(1), \cdots, Z_n^i(d))^T$, $n \geq 0$:

$$\begin{cases} Z_0^i = e_i & \text{(one initial particle of type } i\text{)} \\ Z_{n+1}^i = \sum_{r=1}^{d} \sum_{l=1}^{Z_n^i(r)} N_{l,n}^r & n \geq 0, \end{cases}$$

- $Z_n^i(j) =$ number of particles of type j in generation n;
- $N_{l,n}^r(j) =$ offspring of type j of the l-th particle of type r, of generation n.
- Galton-Watson process : all $N_{l,n}^r$ are independent, and have p.g.f. indep. of n and l : for $s = (s_1, \cdots, s_d) \in [0,1]^d$,

$$f^r(s) = \mathbb{E}\Big(\prod_{j=1}^{d} s_j^{N_{l,n}^r(j)}\Big) = \sum_{k_1,\cdots,k_d=0}^{\infty} p_{k_1,\cdots,k_d}^r s_1^{k_1} \cdots s_d^{k_d},$$

i.e. $\mathbb{P}(N_{l,n}^r = k) = p_k^r, \quad \forall k = (k_1, \cdots, k_d)^T, n \geq 0, l \geq 1.$

Branching process in a random environment

The offspring distributions of gen. n depend on the random environment ξ_n at time n; the random environment sequence $\xi = (\xi_0, \xi_1, \cdots)$ is i.i.d. Denote

$$\mathbb{P}_\xi = \mathbb{P}(\cdot|\xi) \text{ (quenched law)}, \quad \mathbb{E}_\xi = \mathbb{E}[\cdot|\xi]$$

Conditioned on ξ,

- the r.v.'s $N^r_{l,n}$ are independent for $l \geq 1$, $n \geq 0$, $1 \leq r \leq d$;
- each $N^r_{l,n}$, $l \geq 1$ has probability generating function

$$f^r_{\xi_n}(s) = \mathbb{E}_\xi\left(\prod_{j=1}^{d} s_j^{N^r_{l,n}(j)}\right) = \sum_{k_1,\cdots,k_d=0}^{\infty} p^r_{k_1,\cdots,k_d}(\xi_n) s_1^{k_1} \cdots s_d^{k_d}.$$

i.e. $\mathbb{P}_\xi(N^r_{l,n} = k) = p^r_k(\xi_n), \ \forall k = (k_1, \cdots, k_d)^T, n \geq 0, l \geq 1.$

Z_n reduces to the Galton-Watson process if $\xi_n = c$ (const.) $\forall n$.

Lyapunov exponent and LLN for the mean matrices M_n

Consider the mean matrices M_n of the offspring distributions and their products : for $n, k \geq 0$,

$$M_n(i,j) = \mathbb{E}_\xi \left[Z_{n+1}(j) \middle| Z_n = e_i \right] = \frac{\partial f^i_{\xi_n}}{\partial s_j}(1), \quad M_{k,n} := M_k \cdots M_n.$$

Then $\mathbb{E}_\xi Z^i_n(j) = M_{0,n-1}(i,j)$. Assume $\mathbb{E} \log^+ \|M_0\| < +\infty$. The Lyapunov exponent of the mean matrices (M_n) is

$$\gamma = \lim_{n \to +\infty} \frac{1}{n} \mathbb{E} \log \|M_{0,n-1}\| = \inf_{k \geq 1} \frac{1}{k} \mathbb{E} \log \|M_{0,k-1}\|.$$

The following strong law of large numbers has been established by Furstenberg and Kesten (1960) :

$$\lim_{n \to +\infty} \frac{1}{n} \log \|M_{n-1,0}\| = \gamma \quad \mathbb{P}\text{-a.s.}$$

$$\gamma = \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}\log\|M_{0,n-1}\| = \inf_{k\geq 1}\frac{1}{k}\mathbb{E}\log\|M_{0,k-1}\|.$$

Classification

We say that the multi-type branching process (Z_n^i) in the random environment ξ is :

- sub-critical if $\gamma < 0$ $\quad (\|Z_n^i\| \underset{n\to+\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.})$
- critical if $\gamma = 0$ $\quad (\|Z_n^i\| \underset{n\to+\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.})$
- supercritical if $\gamma > 0$ $\quad (\mathbb{P}(\|Z_n^i\| \underset{n\to+\infty}{\longrightarrow} +\infty) > 0.)$

Here we only consider the supercritical regime, i.e. $\gamma > 0$.

For the critical and subcritical cases : see e.g. Vatutin & Dyakonova (2020, 2018), Vatutin & Wachtel (2018), Le Page, Peigné & Pham (2018) for the study of survival probability.

Summary

Kesten-Stigum theorem on the GW process

The famous Kesten-Stigum theorem (1966) on the multi-type Galton-Watson process (constant environment case) gives the exponential increasing rate of the population size via a criterion for the non-degeneracy of the limit of the fundamental branching martingale.

Our work is motivated by getting a full extension of the Kesten-Stigum theorem to the random environment case. We will find exactly when the population size $Z_n^i(j)$ grows like its mean :

$$Z_n^i(j) \approx \mathbb{E}_\xi Z_n^i(j) = M_{0,n-1}(i,j),$$

which implies that the process explodes to $+\infty$ at an exponential rate, and permets us to compare in a precise way a branching process in a random environment with the products of random matrices.

Extending Kesten-Stigum theorem to random environment : long standing problem

Extending the Kesten-Stigum theorem to random environment is a long standing problem. For the single type case (d=1), the problem was solved by Athreya and Karlin (1971, sufficiency), and Tanny (1988, necessity). But for the multi-type case ($d > 1$), it has been open for about 50 years.

Our objective : solve this problem in the typical case, by constructing a suitable martingale which reduces to the fundamental branching martingale in the constant environment case, and by establishing a criterion for the non-degeneracy of its limit.

Applications : this work open ways in establishing other fundamental limit theorems, such as law of large numbers, central limit theorems with Berry-Essen bound, and large deviation results.

Summary

This is the constant environment case : each ξ_n is equal to the same constant. Thus the mean matrices M_n are the same constant (non-random) matrix :

$$M := M_0 = M_1 = \ldots = const.\ matrix$$

Perron-Frobénius theorem

Assume that M is primitive, i.e. there exists $n \geq 0$ such that $M^n > 0$. Then the spectral radius $\rho > 0$ of M is a simple eigenvalue of M, and there exist $u, v \in \mathbb{R}^d$, respectively the unique associated right and left eigenvectors such that :

- $u > 0$ and $v > 0$;
- $\|u\| = 1$ and $\langle u, v \rangle = 1$;
- $Mu = \rho u$ and $Mv^T = \rho v^T$;
- $M^n(i,j) \sim \rho^n u(i) v(j), \quad 1 \leq i,j \leq d.$

Kesten-Stigum theorem on G-W process (1966)

Assume that M is primitive with spectral radius $\rho > 1$. Then there exist r.v.'s $W^i \in [0, \infty)$ such that for all $1 \leq j \leq d$,

$$\frac{Z_n^i(j)}{\mathbb{E} Z_n^i(j)} = \frac{Z_n^i(j)}{M^n(i,j)} \underset{n \to +\infty}{\to} W^i \quad \mathbb{P}\text{-a.s.},$$

or equivalently, $\frac{Z_n^i(j)}{\rho^n u(i) v(j)} \to W^i$ a.s. where $u, v > 0$, $Mu = \rho u$, $Mv^T = \rho v^T$.

Moreover, $\max_{1 \leq i \leq d} \mathbb{P}(W^i = 0) < 1$ (non-degenerate) iff

$$\max_{1 \leq i,j \leq d} \mathbb{E}\left(Z_1^i(j) \log^+ Z_1^i(j)\right) < +\infty. \quad (*)$$

When $(*)$ holds, then for all $1 \leq i \leq d$, $\mathbb{E} W^i = 1$, and a.s.

$$\{W^i = 0\} = \{\|Z_n^i\| \underset{n \to +\infty}{\to} 0\}, \quad \{W^i > 0\} = \{\|Z_n^i\| \underset{n \to +\infty}{\to} +\infty\}.$$

Summary

Condition **FK** (Furstenberg- Kesten)

There exists a constant $C > 1$ such that

$$1 \leq \frac{\max_{1\leq i,j\leq d} M_0(i,j)}{\min_{1\leq i,j\leq d} M_0(i,j)} \leq C \quad \mathbb{P}\text{-a.s.}$$

LLN for the components $M_{0,n-1}(i,j)$ (Furstenberg-Kesten 1960)

Assume condition **FK** and $\mathbb{E} \log^+ \|M_0\| < +\infty$. Then for all $1 \leq i,j \leq d$,

$$\lim_{n\to+\infty} \frac{1}{n} \log M_{0,n-1}(i,j) = \gamma \quad \mathbb{P}\text{-a.s.}$$

KS theorem (Grama-Liu-Pin, Ann. Appl. Prob. 2021+)

Assume condition **FK**, $\gamma > 0$, and $\xi = (\xi_n)_{n\geq 0}$ i.i.d. Then there exist random variables $W^i \in [0, \infty)$ such that for all $1 \leq j \leq d$,

$$\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} = \frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \xrightarrow[n\to+\infty]{\mathbb{P}} W^i$$

Moreover, $\max_{1\leq i\leq d} \mathbb{P}(W^i = 0) < 1$ (W^i non-degenerate) iff

$$\max_{1\leq i,j\leq d} \mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i,j)} \log^+ \frac{Z_1^i(j)}{M_0(i,j)}\right) < +\infty. \quad (*)$$

When $(*)$ holds, then for all $1 \leq i \leq d$, a.s. $\mathbb{E}_\xi W^i = 1$ and

$$\{W^i = 0\} = \{\|Z_n^i\| \underset{n\to+\infty}{\to} 0\}, \quad \{W^i > 0\} = \{\|Z_n^i\| \underset{n\to+\infty}{\to} +\infty\}.$$

A similar version is found for stationary and ergodic environment.

Remarks

1. When $d = 1$, due to Athreya-Karlin (1971) and Tanny (1988)
2. Cohn (Ann. Prob. 1988) claimed the convergence in L^2 of $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}$, under some bounded conditions on the first and second moments of the offspring distribution. But there is a missing quantitative condition in his claim (which is essential even in the single type case $d = 1$).
3. Jones (1997), Biggins, Cohn and Nerman (1999) have studied respectively the L^2 and L^p ($p > 1$) convergence of multi-type branching processes in varying environment. Their results give sufficient conditions for quenched L^p convergence for multi-type branching processes in random environments.

Summary

This is the constant environment case : all ξ_n are equal to the same constant, so that all the mean matrices M_n are the same constant (non-random) matrix :

$$M := M_0 = M_1 = \ldots = const.\ matrix.$$

By the Perron-Frobenius theorem, when M is primitive, the spectral radius $\rho > 0$ of M is a simple eigenvalue of M, with unique right eigenvector $u \in \mathbb{R}^d$ such that :

- $u > 0$;
- $\|u\| = 1$;
- $M\, u = \rho\, u$.

Obviously the last relation, $M\, u = \rho\, u$, is stable for products of M :

$$M^n\, u = \rho^n\, u, \quad n \geq 1.$$

Fundamental Martingale for the Galton-Watson process

Assume that M is primitive with spectral radius $\rho > 1$. Then for all $1 \leq i \leq d$,

$$W_0^i = 1, \quad W_n^i = \frac{\langle Z_n^i, u \rangle}{\rho^n u(i)}, \quad n \geq 1,$$

is a non-negative martingale w.r.t. the filtration

$$\mathcal{F}_n = \sigma\left(N_{l,k}^r(j), 0 \leq k \leq n-1, 1 \leq r, j \leq d, l \geq 1\right),$$

so that $W^i := \lim_{n\to\infty} W_n^i$ exists a.s. with values in $[0, \infty)$.

- The key point in the proof is the fact that $M^n u = \rho^n u$.
- How to extend this to RE is not so clear : one would think of $\frac{\langle Z_n^i, U_{0,n-1}\rangle}{\rho_{0,n-1} U_{0,n-1}}$ or $\frac{\langle Z_n^i, U_{0,n-1}\rangle}{(M_{0,n-1} U_{0,n-1})(i)}$, with $\rho_{0,n-1}$ the spectral radius of $M_{0,n-1}$ and $M_{0,n-1} U_{0,n-1} = \rho_{0,n-1} U_{0,n-1}$, but these are not martingales.

Summary

Consider the products of matrices :

$$M_{n,n+k} = M_n \dots M_{n+k}, \quad n, k \geq 0.$$

Condition H

M_0 is allowable (every row and column contains a strictly positive element) $\mathbb{P}$-a.s., and $\mathbb{P}(\exists k \geq 0 \; M_{0,k} > 0) > 0$.

By the Perron-Frobenius theorem, under condition **H**, the spectral radius $\rho_{n,n+k}$ of $M_{n,n+k}$ is a strictly positive eigenvalue of $M_{n,n+k}$, with right and left eigenvectors $U_{n,n+k}$ and $V_{n,n+k}$ such that :

- $U_{n,n+k} \geq 0 \quad$ and $\quad V_{n,n+k} \geq 0$;
- $\|U_{n,n+k}\| = 1 \quad$ and $\quad \langle U_{n,n+k}, V_{n,n+k} \rangle = 1$;
- $M_{n,n+k} \, U_{n,n+k} = \rho_{n,n+k} \, U_{n,n+k}$;
- $M^T_{n,n+k} \, V_{n,n+k} = \rho_{n,n+k} \, V_{n,n+k}$.

Pseudo spectral radii of pos. random matrices (Hennion 1997)

Assume condition **H**. Then for all $n \geq 0$, a.s.

$$U_{n,\infty} := \lim_{k \to +\infty} U_{n,n+k} > 0 \text{ with } \|U_{n,\infty}| = 1;$$

the scalars $\lambda_n = \|M_n U_{n+1,\infty}\|$ are strictly positive and satisfy

$$M_n U_{n+1,\infty} = \lambda_n U_{n,\infty}. \quad (*)$$

- The sequences $(U_{n,\infty})_{n\geq 0}$ et $(\lambda_n)_{n\geq 0}$ are stationary and ergodic, $U_{n,\infty}$ and λ_n depend only on $\xi_n, \xi_{n+1}, \ldots$
- The numbers λ_n are called pseudo spectral radii of (M_n).
- The relation (*) is stable for products : for all $n, k \geq 0$,

$$M_{n,n+k}\ U_{n+k+1,\infty} = \lambda_{n,n+k}\ U_{n,\infty}, \quad \text{with} \quad \lambda_{n,n+k} := \lambda_n \ldots \lambda_{n+k}.$$

Fundamental martingale for the multi-type branching process in random environment (Grama-Liu-Pin 2021+)

Assume condition **H**. Then for all $1 \leq i \leq d$,

$$W_0^i = 1, \quad W_n^i = \frac{\langle Z_n^i, U_{n,\infty}\rangle}{\lambda_{0,n-1} U_{0,\infty}(i)} = \frac{\langle Z_n^i, U_{n,\infty}\rangle}{(M_{0,n-1} U_{n,\infty})(i)}, \quad n \geq 1.$$

is a non-negative martingale w.r.t. the filtration

$$\mathcal{F}_n = \sigma\left(\xi, N_{l,k}^r(j), 0 \leq k \leq n-1, 1 \leq r, j \leq d, l \geq 1\right),$$

under the laws $\mathbb{P}$ and $\mathbb{P}_\xi$, so that

$$W^i := \lim_{n\to\infty} W_n^i \quad \text{exists a.s. with values in } [0, \infty).$$

- When $\xi_n = \text{const.}$, $W_n^i = \frac{\langle Z_n^i, u\rangle}{(M^n u)(i)} = \frac{\langle Z_n^i, u\rangle}{\rho^n u(i)}$ since $M^n u = \rho^n u$.
- When $d = 1$, $W_n^1 = \frac{Z_n^1}{m_0 \cdots m_{n-1}}$, $m_j = \sum_k k p_k(\xi_j)$.

Proof of the fundamental martingale

$$\mathbb{E}_\xi[Z^i_{n+1}(j)|\mathcal{F}_n] = \sum_{r=1}^{d}\sum_{l=1}^{Z^i_n(r)} \mathbb{E}_\xi N^r_{l,n}(j) = \sum_{r=1}^{d} Z^i_n(r) M_n(r,j) = (M_n^T Z^i_n)(j).$$

$$\begin{aligned}
\mathbb{E}_\xi[W^i_{n+1}|\mathcal{F}_n] &= \frac{\langle \mathbb{E}_\xi[Z^i_{n+1}|\mathcal{F}_n], U_{n+1,\infty}\rangle}{\lambda_{0,n} U_{0,\infty}(i)} \\
&= \frac{\langle M_n^T Z^i_n, U_{n+1,\infty}\rangle}{\lambda_{0,n} U_{0,\infty}(i)} \quad (\text{since} \quad \mathbb{E}_\xi[Z^i_{n+1}|\mathcal{F}_n] = M_n^T Z^i_n) \\
&= \frac{\langle Z^i_n, M_n U_{n+1,\infty}\rangle}{\lambda_{0,n} U_{0,\infty}(i)} \\
&= \frac{\langle Z^i_n, \lambda_n U_{n,\infty}\rangle}{\lambda_{0,n} U_{0,\infty}(i)} \quad (\text{since} \quad M_n U_{n+1,\infty} = \lambda_n U_{n,\infty}) \\
&= \frac{\langle Z^i_n, U_{n,\infty}\rangle}{\lambda_{0,n-1} U_{0,\infty}(i)} \quad (\text{since} \quad \lambda_{0,n} = \lambda_{0,n-1}\lambda_n) \quad = W^i_n.
\end{aligned}$$

Equivalence of the product $\lambda_{0,n-1}$

Assume condition **H**. Then

$$\lambda_{0,n-1} \underset{n\to+\infty}{\sim} \rho_{0,n-1}\langle V_{0,n-1}, U_{n,\infty}\rangle \quad \mathbb{P}\text{-a.s.}$$

LLN for the product $\lambda_{0,n-1}$

Assume condition **H** and $\mathbb{E}\log^+\|M_0\| < +\infty$. Then the expectation $\mathbb{E}\log\lambda_0$ is well defined, and

$$\lim_{n\to+\infty}\frac{1}{n}\log\lambda_{0,n-1} = \mathbb{E}\log\lambda_0 = \gamma \quad \mathbb{P}\text{-a.s.}$$

Applications

The Kesten-Stigum theorem can be applied to establish other very interesting limit theorems such as :

- Law of large numbers and large deviations for $Z_n^i(j)$ and $\|Z_n^i\| = \sum_{j=1}^n Z_n^i(j)$ (total population of gen. n)
- Central limit theorem, Berry-Essen bound and Cramér's moderate deviation expansion for $Z_n^i(j)$, and $\|Z_n^i\|$.

LLN and CLT are immediate consequences : e.g. a.s. on the survival event $S := \{\|Z_n^i\| \to \infty\}$, with $\bar{Z}_n^i(j) = \frac{Z_n^i(j)}{M_{0,n-1}(i,j)}$,

$$\frac{\log Z_n^i(j)}{n} = \frac{\log M_{0,n-1}(i,j)}{n} + \frac{\log \bar{Z}_n^i(j)}{n} \to \gamma,$$

as $\bar{Z}_n^i(j) \to W^i \in (0, \infty)$ a.s. on S.

Berry-Esseen bound and Cramér's LD expansion

For the rates of convergence in the LLN and CLT, a careful analysis is still needed.
For example, using recent results on the products of random matrices by Grama-Liu-Xiao (J. European Math. Soc., 2021+), we obtained (see hal-02911865 and hal-02934081) :

1. Berry-Esseen's bound for $\log \|Z_n^i\|$, which gives the absolute error in the Gaussian approximation :

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} \leq x \right) - \Phi(x) \right| \leq \frac{C}{\sqrt{n}}, \quad \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt.$$

2. Cramér's large deviation expansion , which gives an asymptotic expression of the the relative error in the Gaussian approx. :

$$\frac{\mathbb{P}\left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} > x \right)}{1 - \Phi(x)} = \cdots \text{ (asym. expression)}, \quad 0 \leq x = o(\sqrt{n}).$$

More results can be found in the thesis by Erwan Pin (2020).

References

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Thanks for your attention !

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